1.5 Solutions Sets of Linear Systems

Consider the following exercises

#2 Determine if the system has a nontrivial ($\vec{x} \neq \vec{0}$) solution.

$$
\begin{aligned}
x_1 - 2x_2 + 3x_3 &= 0 \\
-2x_1 - 3x_2 - 4x_3 &= 0 \\
2x_1 - 4x_2 + 9x_3 &= 0
\end{aligned}
\qquad
\begin{pmatrix} 1 & -2 & 3 & 0 \\ -2 & -3 & -4 & 0 \\ 2 & -4 & 9 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 3 & 0 \\ 0 & -7 & 2 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}
$$

Then, $3x_3 = 0 \Rightarrow x_3 = 0$

$$-7x_2 + 2x_3 = 0 \Rightarrow -7x_2 = 0 \Rightarrow x_2 = 0$$

$$-x_1 - x_2 + 3x_3 = 0 \Rightarrow -x_1 = 0 \Rightarrow x_1 = 0$$

Only solution is the trivial solution.

Another way to look at this problem is to observe that for the echelon form of the coefficient matrix

$$
\begin{pmatrix} 1 & -2 & 3 \\ -2 & -3 & -4 \\ 2 & -4 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 3 \\ 0 & -7 & 2 \\ 0 & 0 & 3 \end{pmatrix}
$$

every column is a pivot column.

Thus, the original linear system does not have a free variable.

Exercise # 5

$$2x_1 + 2x_2 + 4x_3 = 0$$
$$-4x_1 - 4x_2 - 8x_3 = 0$$
$$-3x_2 - 3x_3 = 0$$

$$\begin{pmatrix} 2 & 2 & 4 & 0 \\ -4 & -4 & -8 & 0 \\ 0 & -3 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 0 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_1 + x_3 = 0 \quad \Rightarrow \quad x_1 = -x_3$$
$$x_2 + x_3 = 0 \qquad x_2 = -x_3$$
$$x_3 \text{ free}$$

Then, we can express all the solutions as the set of all vectors $\vec{x}$ such that

$$\vec{x} = \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

This set of vectors is also called "the general solution" of the original system.

<u>Observation</u>: Since this system has a free variable then there are infinitely many solutions.

Exercise #17

$$\begin{aligned} 2x_1 + 2x_2 + 4x_3 &= 8 \\ -4x_1 - 4x_2 - 8x_3 &= -16 \\ -3x_2 - 3x_3 &= 12 \end{aligned}$$

$$\begin{pmatrix} 2 & 2 & 4 & 8 \\ -4 & -4 & -8 & -16 \\ 0 & -3 & -3 & 12 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & -3 & 12 \end{pmatrix} \sim$$

$$\sim \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & -3 & -3 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 & 8 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} x_1 \quad + x_3 &= 8 \\ x_2 + x_3 &= -4 \\ x_3 \text{ free} \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= 8 - x_3 \\ x_2 &= -4 - x_3 \\ x_3 &\text{ free} \end{aligned}$$

We can express all the solutions as the set of vectors $\vec{x}$ such that

$$\vec{x} = \begin{pmatrix} 8 - x_3 \\ -4 - x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_3 \\ -x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

This constitutes the general solution of the original system.

General Theory

Definitions:

a) A system of the form $A\vec{x}=\vec{0}$ is calle "homogeneous"

b) " " " " " $A\vec{x}=\vec{b}$, $\vec{b}\neq 0$ is called "nonhomogeneous".

c) If $\vec{x}=\vec{0}$ is a solution for a linear system $A\vec{x}=\vec{0}$, this solution is called "the trivial solution". Any other solution is called "nontrivial solution".

Thm. The homogeneous equation $A\vec{x}=\vec{0}$ has a nontrivial solution(*) if and only if the equation has at least one free variable.

Statement (p)

Statement (q)

Go back to the solutions of exercises #2 and #5.

(*) Actually has infinitely many solutions.

Proof.- Use backward process to solve the system in both cases: i) Free variable(s).

ii) Not free variables

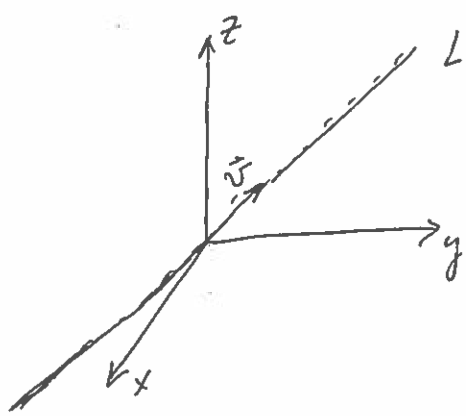
Solutions in parametric vector form and geometric interpretation.

The solutions of the system in exercise #5 can be expressed as

$$\vec{x} = x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

if we call $\vec{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$, then we can say that the set of solns. of this system is given by the line through the origin in the space defined by

$$\{\vec{x} \text{ in } \mathbb{R}^3 \text{ such that } \vec{x} = t\vec{v},\ t \text{ is real}\}.$$



Similarly the solution of exerc. #17 is another line in the space that is the translation of the one above.

In fact, Solns of #17 are

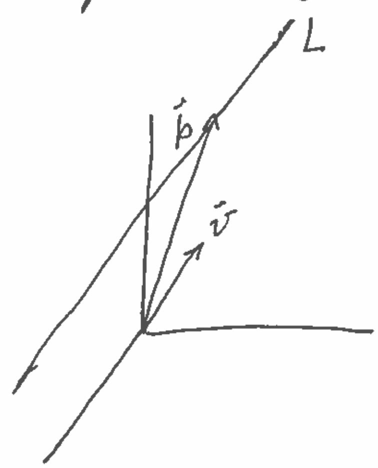
$$\vec{x} = \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

Calling $\vec{p} = \begin{pmatrix} 8 \\ -4 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$ and $t = x_3$

the set of solutions of ex. #17 is given by

$$\left\{ \vec{x} \text{ in } \mathbb{R}^3, \text{ such that } \vec{x} = \vec{p} + t\vec{v}, \ t \text{ is real} \right\}$$

This is a line parallel to $\vec{v}$ passing through the terminal point of $\vec{p}$.



Discuss this in 2-D.

The previous problems motivates the formulation of the following thm.

Thm 6.- 1) $A\vec{x}=\vec{b}$ is consistent, $\vec{b}\neq\vec{0}$.

2) Let $\vec{p}$ be a particular Solution of $A\vec{x}=\vec{b}$.

then, the "general Solution" of $A\vec{x}=\vec{b}$, or the set of all solutions of $A\vec{x}=\vec{b}$ is given by

$$S=\left\{\begin{array}{l}\vec{w} \text{ in } \mathbb{R}^3, \text{ Such that } \vec{w}=\vec{p}+\vec{v}_h, \text{ where } \vec{v}_h \text{ is any Soln.} \\ \text{of the corresponding homogeneous equation}\end{array}\right\}$$

Proof.- Exercise # 25. Pay attention to this problem.

Some Conclusions are:

i) We recover here thm 2

If $A\vec{x}=\vec{b}$ is consistent and it has no free variables then $\vec{v}_h=\vec{0}$, and the only soln. is $\vec{x}=\vec{p}$.

ii) If there are free variables then there are infinitely many Solutions for both systems, and the solns. for $A\vec{x}=\vec{b}$ can be obtained translating by the vector $\vec{p}$, the Solutions of $A\vec{x}=\vec{0}$.

Solutions of a Single equation with 3 Variables in parametric form.

$$x_1 + 5x_2 - 3x_3 = 0$$

$x_1 = -5x_2 + 3x_3$
x_2 free $\Rightarrow$ Set of Solns $\vec{x} = \begin{pmatrix} -5x_2+3x_3 \\ x_2 \\ x_3 \end{pmatrix} =$
x_3 free

$$= \begin{pmatrix} -5x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 3x_3 \\ 0 \\ x_3 \end{pmatrix} = x_2 \underbrace{\begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix}}_{=\vec{u}} + x_3 \underbrace{\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}}_{=\vec{v}}$$

or

$$\vec{x} = s\vec{u} + t\vec{v}, \qquad s, t \text{ real numbers.}$$

Therefore, Set of Solutions is given by all vectors Lying in a plane generated by the two vectors $\vec{u}$ and $\vec{v}$.

Thm 6.- Hypothesis

1) $A\vec{x}=\vec{b}$ is consistent for a given $\vec{b}$

2) $\vec{p}$ is a solution. It means $A\vec{p}=\vec{b}$.

Then

1) Any solution $\vec{x}^*$ of $A\vec{x}=\vec{b}$ can be written as

$$\vec{x}^* = \vec{p} + \vec{V}_h^*,$$ where $\vec{V}_h^*$ is a solution of $A\vec{x}=\vec{0}$ i.e., $A\vec{v}_h^*=0$

2) The set

$$S = \{ \vec{w} \text{ in } \mathbb{R}^n, \text{ such that } \vec{w} = \vec{p}+\vec{V}_h, \ \vec{V}_h \text{ is any solution of } A\vec{x}=\vec{0} \}$$

is the set of all solutions of $A\vec{x}=\vec{b}$.

Section 1.5 Exercise #25

Proof.- 1) Let $\vec{x}^*$ be a solution of $A\vec{x}=\vec{b}$, then $A\vec{x}^*=\vec{b}$

we also know that $A\vec{p}=\vec{b}$.

Then, $$A(\vec{x}^*-\vec{p}) = A\vec{x}^* - A\vec{p} = \vec{b}-\vec{b} = \vec{0}$$

Therefore, $\vec{x}^*-\vec{p}$ is a solution of $A\vec{x}=\vec{0}$

by defining $$\vec{V}_h^* = \vec{x}^* - \vec{p}$$

we find $\vec{x}^* = \vec{p} + \vec{V}_h^*$, where $\vec{V}_h^*$ is a soln. of $A\vec{x}=\vec{0}$.

This means $\vec{x}^*$ is in S.

2) We want to prove

$$T = \left\{ \begin{array}{c} \text{Set of} \\ \text{Solns} \\ \text{of } A\vec{x} = \vec{b} \end{array} \right\} \subset S \quad \text{and} \quad \overset{\text{(B)}}{S \subset T} = \left\{ \begin{array}{c} \text{Set of} \\ \text{Solns} \\ \text{of } A\vec{x} = \vec{b} \end{array} \right\}$$

Proof of (A) was given by part (1)

Any soln. of $A\vec{x} = \vec{b}$ is in S.

It means if $\vec{x} \in T \Rightarrow \vec{x} \in S \Rightarrow T \subset S$

Proof of (B)

If $\vec{w}$ is in S, then $\vec{w} = \vec{p} + \vec{v}_h$, where $\vec{v}_h$ is a soln. of $A\vec{x} = \vec{0}$.

Now, $A\vec{w} = A(\vec{p} + \vec{v}_h) = A\vec{p} + A\vec{v}_h = \vec{b} + \vec{0} = \vec{b}$

therefore, $\vec{w}$ is also in T

It means if $\vec{w} \in S \Rightarrow \vec{w} \in T \Rightarrow S \subset T$.